theorem can be formulated to establish the existence of a unique **positive nth root** of a, denoted by $\sqrt[n]{a}$ or $a^{1/n}$, for each $n \in \mathbb{N}$.

Remark If in the proof of Theorem 2.4.7 we replace the set S by the set of rational numbers $T := \{r \in \mathbb{Q} : 0 \le r,\ r^2 < 2\}$, the argument then gives the conclusion that $y := \sup T$ satisfies $y^2 = 2$. Since we have seen in Theorem 2.1.4 that y cannot be a rational number, it follows that the set T that consists of rational numbers does not have a supremum belonging to the set $\mathbb{Q}$. Thus the ordered field $\mathbb{Q}$ of rational numbers does *not* possess the Completeness Property.

Density of Rational Numbers in $\mathbb{R}$

We now know that there exists at least one irrational real number, namely $\sqrt{2}$. Actually there are "more" irrational numbers than rational numbers in the sense that the set of rational numbers is countable (as shown in Section 1.3), while the set of irrational numbers is uncountable (see Section 2.5). However, we next show that in spite of this apparent disparity, the set of rational numbers is "dense" in $\mathbb{R}$ in the sense that given any two real numbers there is a rational number between them (in fact, there are infinitely many such rational numbers).

2.4.8 The Density Theorem *If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. It is no loss of generality (why?) to assume that $x > 0$. Since $y - x > 0$, it follows from Corollary 2.4.5 that there exists $n \in \mathbb{N}$ such that $1/n < y - x$. Therefore, we have $nx + 1 < ny$. If we apply Corollary 2.4.6 to $nx > 0$, we obtain $m \in \mathbb{N}$ with $m - 1 \le nx < m$. Therefore, $m \le nx + 1 < ny$, whence $nx < m < ny$. Thus, the rational number $r := m/n$ satisfies $x < r < y$. Q.E.D.

To round out the discussion of the interlacing of rational and irrational numbers, we have the same "betweenness property" for the set of irrational numbers.

2.4.9 Corollary *If x and y are real numbers with $x < y$, then there exists an irrational number z such that $x < z < y$.*

Proof. If we apply the Density Theorem 2.4.8 to the real numbers $x/\sqrt{2}$ and $y/\sqrt{2}$, we obtain a rational number $r \ne 0$ (why?) such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

Then $z := r\sqrt{2}$ is irrational (why?) and satisfies $x < z < y$. Q.E.D.

Exercises for Section 2.4

1. Show that $\sup\{1 - 1/n : n \in \mathbb{N}\} = 1$.
2. If $S := \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.
3. Let $S \subseteq \mathbb{R}$ be nonempty. Prove that if a number u in $\mathbb{R}$ has the properties: (i) for every $n \in \mathbb{N}$ the number $u - 1/n$ is not an upper bound of S, and (ii) for every number $n \in \mathbb{N}$ the number $u + 1/n$ is an upper bound of S, then $u = \sup S$. (This is the converse of Exercise 2.3.9.)

4. Let S be a nonempty bounded set in $\mathbb{R}$.
   (a) Let $a > 0$, and let $aS := \{as : s \in S\}$. Prove that
$$\inf(aS) = a \inf S, \quad \sup(aS) = a \sup S.$$
   (b) Let $b < 0$ and let $bS = \{bs : s \in S\}$. Prove that
$$\inf(bS) = b \sup S, \quad \sup(bS) = b \inf S.$$

5. Let S be a set of nonnegative real numbers that is bounded above and let $T := \{x^2 : x \in S\}$. Prove that if $u = \sup S$, then $u^2 = \sup T$. Give an example that shows the conclusion may be false if the restriction against negative numbers is removed.

6. Let X be a nonempty set and let $f : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, show that Example 2.4.1(a) implies that
$$\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}.$$
Show that we also have
$$\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}.$$

7. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.

8. Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that
$$\sup\{f(x) + g(x) : x \in X\} \le \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$
and that
$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \le \inf\{f(x) + g(x) : x \in X\}.$$
Give examples to show that each of these inequalities can be either equalities or strict inequalities.

9. Let $X = Y := \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) := 2x + y$.
   (a) For each $x \in X$, find $f(x) := \sup\{h(x, y) : y \in Y\}$; then find $\inf\{f(x) : x \in X\}$.
   (b) For each $y \in Y$, find $g(y) := \inf\{h(x, y) : x \in X\}$; then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (a).

10. Perform the computations in (a) and (b) of the preceding exercise for the function $h : X \times Y \to \mathbb{R}$ defined by
$$h(x, y) := \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x \ge y. \end{cases}$$

11. Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be defined by
$$f(x) := \sup\{h(x, y) : y \in Y\}, \quad g(y) := \inf\{h(x, y) : x \in X\}.$$
Prove that
$$\sup\{g(y) : y \in Y\} \le \inf\{f(x) : x \in X\}.$$
We sometimes express this by writing
$$\sup_y \inf_x h(x, y) \le \inf_x \sup_y h(x, y).$$
Note that Exercises 9 and 10 show that the inequality may be either an equality or a strict inequality.

12. Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $F : X \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ be defined by
$$F(x) := \sup\{h(x,y) : y \in Y\}, \quad G(y) := \sup\{h(x,y) : x \in X\}.$$
Establish the **Principle of the Iterated Suprema:**
$$\sup\{h(x,y) : x \in X, y \in Y\} = \sup\{F(x) : x \in X\} = \sup\{G(y) : y \in Y\}$$
We sometimes express this in symbols by
$$\sup_{x,y} h(x,y) = \sup_x \sup_y h(x,y) = \sup_y \sup_x h(x,y).$$
13. Given any $x \in \mathbb{R}$, show that there exists a *unique* $n \in \mathbb{Z}$ such that $n - 1 \leq x < n$.
14. If $y > 0$, show that there exists $n \in \mathbb{N}$ such that $1/2^n < y$.
15. Modify the argument in Theorem 2.4.7 to show that there exists a positive real number y such that $y^2 = 3$.
16. Modify the argument in Theorem 2.4.7 to show that if $a > 0$, then there exists a positive real number z such that $z^2 = a$.
17. Modify the argument in Theorem 2.4.7 to show that there exists a positive real number u such that $u^3 = 2$.
18. Complete the proof of the Density Theorem 2.4.8 by removing the assumption that $x > 0$.
19. If $u > 0$ is any real number and $x < y$, show that there exists a rational number r such that $x < ru < y$. (Hence the set $\{ru : r \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.)

Section 2.5 Intervals

The Order Relation on $\mathbb{R}$ determines a natural collection of subsets called "intervals." The notations and terminology for these special sets will be familiar from earlier courses. If $a, b \in \mathbb{R}$ satisfy $a < b$, then the **open interval** determined by a and b is the set
$$(a,b) := \{x \in \mathbb{R} : a < x < b\}.$$
The points a and b are called the **endpoints** of the interval; however, the endpoints are not included in an open interval. If both endpoints are adjoined to this open interval, then we obtain the **closed interval** determined by a and b; namely, the set
$$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$
The two **half-open** (or **half-closed**) intervals determined by a and b are $[a, b)$, which includes the endpoint a, and $(a, b]$, which includes the endpoint b.

Each of these four intervals is bounded and has **length** defined by $b - a$. If $a = b$, the corresponding open interval is the empty set $(a, a) = \emptyset$, whereas the corresponding closed interval is the singleton set $[a, a] = \{a\}$.

There are five types of unbounded intervals for which the symbols ∞ (or $+\infty$) and $-\infty$ are used as notational convenience in place of the endpoints. The **infinite open intervals** are the sets of the form
$$(a, \infty) := \{x \in \mathbb{R} : x > a\} \quad \text{and} \quad (-\infty, b) := \{x \in \mathbb{R} : x < b\}.$$

If $c > 1$, then $c^{1/n} = 1 + d_n$ for some $d_n > 0$. Hence by Bernoulli's Inequality 2.1.13(c),

$$c = (1 + d_n)^n \geq 1 + nd_n \quad \text{for} \quad n \in \mathbb{N}.$$

Therefore we have $c - 1 \geq nd_n$, so that $d_n \leq (c-1)/n$. Consequently we have

$$|c^{1/n} - 1| = d_n \leq (c - 1)\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}.$$

We now invoke Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $c > 1$.

Now suppose that $0 < c < 1$; then $c^{1/n} = 1/(1 + h_n)$ for some $h_n > 0$. Hence Bernoulli's Inequality implies that

$$c = \frac{1}{(1 + h_n)^n} \leq \frac{1}{1 + nh_n} < \frac{1}{nh_n},$$

from which it follows that $0 < h_n < 1/nc$ for $n \in \mathbb{N}$. Therefore we have

$$0 < 1 - c^{1/n} = \frac{h_n}{1 + h_n} < h_n < \frac{1}{nc}$$

so that

$$|c^{1/n} - 1| < \left(\frac{1}{c}\right)\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}.$$

We now apply Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $0 < c < 1$.

(d) $\lim(n^{1/n}) = 1$

Since $n^{1/n} > 1$ for $n > 1$, we can write $n^{1/n} = 1 + k_n$ for some $k_n > 0$ when $n > 1$. Hence $n = (1 + k_n)^n$ for $n > 1$. By the Binomial Theorem, if $n > 1$ we have

$$n = 1 + nk_n + \tfrac{1}{2}n(n-1)k_n^2 + \cdots \geq 1 + \tfrac{1}{2}n(n-1)k_n^2,$$

whence it follows that

$$n - 1 \geq \tfrac{1}{2}n(n-1)k_n^2.$$

Hence $k_n^2 \leq 2/n$ for $n > 1$. If $\varepsilon > 0$ is given, it follows from the Archimedean Property that there exists a natural number N_ε such that $2/N_\varepsilon < \varepsilon^2$. It follows that if $n \geq \sup\{2, N_\varepsilon\}$ then $2/n < \varepsilon^2$, whence

$$0 < n^{1/n} - 1 = k_n \leq (2/n)^{1/2} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\lim(n^{1/n}) = 1$. □

Exercises for Section 3.1

1. The sequence (x_n) is defined by the following formulas for the nth term. Write the first five terms in each case:

(a) $x_n := 1 + (-1)^n$, (b) $x_n := (-1)^n/n$,

(c) $x_n := \dfrac{1}{n(n+1)}$, (d) $x := \dfrac{1}{n^2 + 2}$.

2. The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term x_n.
   (a) $5, 7, 9, 11, \ldots,$ (b) $1/2, -1/4, 1/8, -1/16, \ldots,$
   (c) $1/2, 2/3, 3/4, 4/5, \ldots,$ (d) $1, 4, 9, 16, \ldots.$
3. List the first five terms of the following inductively defined sequences.
   (a) $x_1 := 1, \quad x_{n+1} := 3x_n + 1,$
   (b) $y_1 := 2, \quad y_{n+1} := \frac{1}{2}(y_n + 2/y_n),$
   (c) $z_1 := 1, \quad z_2 := 2, \ z_{n+2} := (z_{n+1} + z_n)/(z_{n+1} - z_n),$
   (d) $s_1 := 3, \quad s_2 := 5, \ s_{n+2} := s_n + s_{n+1}.$
4. For any $b \in \mathbb{R}$, prove that $\lim(b/n) = 0$.
5. Use the definition of the limit of a sequence to establish the following limits.
   (a) $\lim\left(\dfrac{n}{n^2+1}\right) = 0,$ (b) $\lim\left(\dfrac{2n}{n+1}\right) = 2,$
   (c) $\lim\left(\dfrac{3n+1}{2n+5}\right) = \dfrac{3}{2},$ (d) $\lim\left(\dfrac{n^2-1}{2n^2+3}\right) = \dfrac{1}{2}.$
6. Show that
   (a) $\lim\left(\dfrac{1}{\sqrt{n+7}}\right) = 0,$ (b) $\lim\left(\dfrac{2n}{n+2}\right) = 2,$
   (c) $\lim\left(\dfrac{\sqrt{n}}{n+1}\right) = 0,$ (d) $\lim\left(\dfrac{(-1)^n n}{n^2+1}\right) = 0.$
7. Let $x_n := 1/\ln(n+1)$ for $n \in \mathbb{N}$.
   (a) Use the definition of limit to show that $\lim(x_n) = 0$.
   (b) Find a specific value of $K(\varepsilon)$ as required in the definition of limit for each of (i) $\varepsilon = 1/2$, and (ii) $\varepsilon = 1/10$.
8. Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$. Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).
9. Show that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim(x_n) = 0$, then $\lim(\sqrt{x_n}) = 0$.
10. Prove that if $\lim(x_n) = x$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.
11. Show that $\lim\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) = 0.$
12. Show that $\lim(\sqrt{n^2+1} - n) = 0$.
13. Show that $\lim(1/3^n) = 0$.
14. Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$. [*Hint*: Use the Binomial Theorem as in Example 3.1.11(d).]
15. Show that $\lim\left((2n)^{1/n}\right) = 1$.
16. Show that $\lim(n^2/n!) = 0$.
17. Show that $\lim(2^n/n!) = 0$. [*Hint*: If $n \geq 3$, then $0 < 2^n/n! \leq 2\left(\frac{2}{3}\right)^{n-2}$.]
18. If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.